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Is Non-Ontological Structuralism Hypothetical?

CZY STRUKTURALIZM NIEONTOLOGICZNY JEST HIPOTETYCZNY?

Streszczenie

Michael Resnik, twórca nowoczesnego strukturalizmu w filozofii matematyki, na pewnym etapie swojej twórczości naukowej zmienił poglądy i zaproponował nowy strukturalizm nieontologiczny. Resnik uważany jest za wybitną postać współczesnego strukturalizmu w obszarze współczesnej filozofii matematyki, a jego strukturalizm *sui generis* jest uważany za jedno z najważniejszych i najczęściej dyskutowanych stanowisk w tej dziedzinie. W artykule zbadano motywacje stojące za zmianą poglądów Resnika. Zostało szczegółowo zaprezentowane jego stanowisko i podjęto próbę skontrastowania tego stanowiska z wybranymi poglądami z zakresu filozofii matematyki. Niniejszy wywód nawiązuje do sporu Gottloba Fregego z Davidem Hilbertem, który koncentruje się na statusie aksjomatów teorii matematycznej oraz znaczeniu przypisywanym terminom pierwotnym. Wspomniano także trzyetapową koncepcję rozwoju nauk dedukcyjnych zaproponowaną przez Kazimierza Ajdukiewicza. Koncepcja ta, inspirowana ideami Hilberta, wyznacza trzy etapy ewolucji teorii dedukcyjnych: (1) dedukcja przedaksjomatyczna, (2) dedukcja aksjomatyczna, (3) aksjomatyka abstrakcyjna. Każdy z tych etapów ma unikalne cechy, które rzucają światło na naturę teorii dedukcyjnych, szczególnie w odniesieniu do zbiorów aksjomatów i terminów pierwotnych w ramach tych teorii. Dodatkowo omówiono dwa style uprawiania teorii dedukcyjnych (asertywny i hipotetyczny). Ostatecznie badanie nieontologicznego strukturalizmu Resnika dostarcza wglądu w to, jak należy rozumieć tę nowatorską koncepcję strukturalizmu, i wyjaśnia faktyczne twierdzenia autora. Oprócz tych rozróżnień sformułowana jest nowa koncepcja hipotetycznego strukturalizmu, odrębna od strukturalizmu nieontologicznego. Ten nowatorski strukturalizm jest zakorzeniony w hipotetycznym podejściu do praktykowania teorii dedukcyjnych.

Słowa kluczowe: filozofia matematyki, Resnik, strukturalizm nieontologiczny, strukturalizm hipotetyczny

Introduction

The concept of structuralism within the philosophy of mathematics is still being developed and vigorously debated. The best example of this is the position of non-ontological structuralism that Michael Resnik recently proposed.[1] Among the many different concepts that fall within structuralist thought, this one is noteworthy at least because Resnik is known as a protagonist of *sui generis* structuralism. The shifting views of this author, while remaining in the structuralist position, is an interesting phenomenon deserving a deeper analysis.

Resnik's original concept was presented and developed in a series of works.[2] Many authors, as well as Resnik himself, classify this version of his views as structuralism *sui generis*,[3] which assumes that the objects studied by mathematics are the structures and positions in these structures, and the structures are treated as existing in a specific, definite way, most frequently as abstract entities. As such, this standpoint is sometimes also classified as non-eliminative structuralism.[4]

The concept of *sui generis* structuralism is also compared to the position of conceptual realism within the dispute over the universals. Variants of this position maintain that general concepts exist as abstract entities in one way or another, on the same principle that structures exist as mathematical objects. In some ways, this assertion of existence can be regarded as an ontologically positive position. Other positions within structuralism itself are also discussed, especially the concept of eliminative structuralism,[5] as opposed to Resnik's old views. In this approach, mathematical structures are not considered abstract entities to which existence is attributed. Eliminative concepts are compared with the nominalist position within the dispute over the universals and can thus be treated as ontologically negative.

1 M. Resnik, *Non-ontological Structuralism*, "Structural Relativity" 27 (2019) 3, pp. 303–315.

2 Idem, *Mathematics as a Science of Patterns: Ontology and Reference*, "Noûs" 15 (1981) 4, pp. 529–550; idem, *Mathematics as a Science of Patterns: Epistemology*, "Noûs" 16 (1982) 1, pp. 95–105; idem, *Mathematics from the Structural Point of View*, "Revue Internationale de Philosophie" 42 (1988) 167, pp. 400–424; idem, *Structural Relativity*, "Structural Relativity" 4 (1996) 2, pp. 83–99; idem, *Mathematics as a Science of Patterns*, Oxford 1997.

3 Cf. G. Hellman, *Three Varieties of Mathematical Structuralism*, "Philosophia Mathematica" 9 (2001) 2, pp. 184–211; M. Resnik, *Non-ontological Structuralism*…, pp. 303–315.

4 E. Reck, G. Schiemer, *Structuralism in the Philosophy of Mathematics*, in: *Stanford Encyclopedia in Philosophy* (SEP) [accessed: 11.09.2020].

5 Cf. ibidem.

This article focuses on an attempt to explain Resnik's proposal of non-ontological structuralism. We aim to understand the motivations and actual views that Resnik is currently proposing. Throughout the discussion, this position will be contrasted with other views so as to finally answer the title question of whether, according to non-ontological structuralism, the claims of mathematical theories are mere hypotheses.

1. Non-ontological structuralism

Resnik, arguing for non-ontological structuralism, shows that this position is a new, original view. He believes that categorically advocating the existence or non-existence of mathematical structures, and therefore their nature, is the incorrect approach, and as such, this issue should be left undecided.[6] This is a fundamental change in this author's views, which is particularly interesting because, according to him, the recognition of structures as objects studied by mathematics still stands. Thus, despite the change in views, mathematical structures remain at the centre of interest.

Resnik moved from the position of *sui generis* structuralism, in which he considers structures (called patterns) as existing abstract entities, to the position of non-ontological structuralism, which suspends judgment on the existence and nature of structures. Based on the first position, we can state that structures are objects studied by mathematics, which can be described as correct or incorrect (true or false).[7] Basic mathematical entities, such as numbers or points in this case, have no properties beyond structure (non-relational properties), and even if they did, they are of no interest to mathematicians. In this sense, mathematical structures are abstract objects in which only certain relational properties between places (points) are captured by the structures, while possible other properties of the objects occupying these places are irrelevant.[8]

The main motivation for changing this view is the vague observation that it contains "too much ontology,"[9] so Resnik proposes a new position of non-ontological structuralism. The main difference between the old and new approaches is the way the subject of mathematics is described. From Resnik's new point of view, he argues that when practising the philosophy of mathematics, one should focus more on mathematical theories and not on the objects studied by mathematics. According to him, mathematicians are primarily concerned with the development of theories,

6 M. Resnik, *Non-ontological Structuralism…*, p. 303.

7 Cf. idem, *Mathematics as a Science of Patterns…*, (1997), p. 201.

8 Cf. ibidem, pp. 202–203.

9 M. Resnik, *Non-ontological Structuralism…*, p. 309.

which can only secondarily be seen as saying something about mathematical objects, including structures of accounts.

> So instead of expressing my view by putting the emphasis on objects, I will put the emphasis on theories: Mathematics speaks of objects in order to describe or present structures; from the point of view of a mathematical theory, the denotations of its constants and quantifiers might as well be points or positions in a structure or structures; for the theory attributes to them no identifying features outside of the structure or structures in question.[10]

According to Resnik, this change is merely a change in the ontological attitude regarding mathematical objects. Resnik, inspired by Willard Van Orman Quine's concept of global structuralism,[11] instead of talking about positions in structures and the related ways they exist, aims to discuss the terms we insert in place of variables in mathematical formulas and what is subject to quantification within mathematical theories.

> Much of my old view survives. Now, instead of talking about positions in patterns, we talk about theories and singular terms and quantifiers. Instead of saying that there is no fact as to whether the positions of a natural number sequence are identical to a certain sets, we say that there is no fact as to which of the many interpretations of number theory in set theory is the correct one. This is just a consequence of ontological relativity without the explanation in terms of positions in patterns.[12]

His new concept is structuralist in the sense that he still contributes to discourse about structures as objects studied by mathematics, but at the same time he believes that mathematical theories are the right tool for describing this reality, and only they are subject to study by mathematics.[13]

2. The conception of the development of mathematics

Resnik's position is clearer if we pay attention to David Hilbert's structuralist views, which provide some inspiration for non-ontological structuralism. Hilbert's philosophical views relating to the nature of objects described by mathematical axioms are interpreted today as structuralist. The main source for these interpretations is the correspondence between Hilbert and Gottlob Frege in the early 20th century. The discussion between these two authors today is known as the Frege-Hilbert Controversy. In this context, it is appropriate to speak of the influence of Hilbert's views,

10 Ibidem.

11 Ibidem, pp. 306–308.

12 Ibidem, p. 310.

13 Ibidem.

which led to the formation of modern structuralist positions, rather than to classify Hilbert as a structuralist, as Fiona T. Doherty does,[14]

> The Frege-Hilbert Controversy has been the subject of much analysis and discussion.[15] The main axis of the dispute is disagreement over the meaning of the primary terms of axiomatic theories, especially mathematical ones. In the most general terms, Hilbert claimed that the meaning of primary terms is determined by the axioms and theorems of the theory, while Frege maintains that the meaning of these terms was previously known before the axioms were formulated.

Hilbert wrote in a letter to Frege in December 1899:

> …it is surely obvious that every theory is only a scaffolding or schema of concepts together with their necessary relations to one another, and that the basic elements can be thought of in any way one likes. If in speaking of my points I think of some system of things, e.g., the system: love, law, chimney-sweep […] and then assume all my axioms as relations between these things, then my propositions, e.g., Pythagoras' theorem, are also valid for these things.[16]

Hilbert allows for multiple possible interpretations of one theory. In this way, he can explicitly say that a theory is simply a scheme or scaffolding. This statement is often interpreted as proof that Hilbert was a structuralist, especially when the concept of a schema or scaffolding is understood as the modern concept of mathematical structure.[17]

However, in Frege's conception, the axioms of mathematical theories are true sentences because they refer to specific objects that are indicated by the primary terms.[18] Thus, the source of the truthfulness of mathematical axioms is the meaning that is attributed to the primary terms of the mathematical theory. On the other hand, the consistency of a mathematical theory does not require any special proof. It is different in Hilbert's conceptualization, as he believes that primary terms have no other meaning outside the theory. Hence, mathematical axioms cannot be true in this particular sense, and the proof of the consistency of a theory is the central

14 F. Doherty, *Hilbertian Structuralism and the Frege-Hilbert Controversy*, "Philosophia Mathematica" 27 (2019) 3, pp. 335–361.

15 M. Resnik, *The Frege-Hilbert Controversy*, "Philosophy and Phenomenological Research" 34 (1974) 3, pp. 386–403; S. Shapiro, *Categories, Structures, and the Frege-Hilbert Controversy: The Status of Meta-mathematics*, "Philosophia Mathematica" 13 (2005) 3, pp. 61–77; F. Doherty, *Hilbertian Structuralism…*, pp. 335–361.

16 G. Gabriel et al. (eds.), *Gottlob Frege: Philosophical and Mathematical Correspondence*, Oxford 1980, pp. 40–41.

17 Cf. F. Doherty, *Hilbertian Structuralism…*, p. 338.

18 Cf. S. Shapiro, *Categories, Structures, and the Frege-Hilbert Controversy…*, p. 66.

task for its creator. Hilbert's somewhat obscure view was creatively developed and presented by his student, Kazimierz Ajdukiewicz.

Ajdukiewicz's concept can be understood as an idealized description of the methodological development of mathematical theory. The first works on this subject were written as early as 1921,[19] while a textbook elaboration of them can be found in *Pragmatic Logic*.[20] These views are also actively discussed today.[21]

According to Ajdukiewicz, three stages of the development of deductive sciences can be distinguished. The first is pre-axiomatic intuitive, the second is axiomatic intuitive, and the third is axiomatic abstract.

The first stage, the pre-axiomatic intuitive, is characterized by the fact that the axioms and the primary terms used in them are taken as obvious and intuitively understood, and thus, true. The sets of both axioms and primary terms are not fixed definitively and can be expanded at any time with a new element. The decision to expand the theory with an additional axiom or new term is made only on the condition of absolute obviousness and the possible absence of opposition from other researchers. Axioms are treated as true sentences because of their content, which is intuitively understandable, due to the primary terms used within them. Importantly, this stage is also characterized by the fact that the proof procedure is based on obviousness and intuition, so there is no closed set of acceptable rules for the proof procedure. Those theorems whose proofs are considered correct by intuition are considered proven. The essential features of this stage are intuitiveness and openness.

The second stage, or the axiomatic intuitive stage, is distinguished from the first primarily by the fact that sets of axioms and the primary terms used in them are established and closed. In this stage, no axiom, even the most obvious one, can be freely added if it has not been indicated at the beginning that it belongs to the theory. The same is true of the primary terms, the set of these terms should be fixed and closed once and for all. All theorems of the theory use only primary terms or can be reduced to them. It is not acceptable to use terms other than those adopted at the beginning. On the other hand, the method of establishing both the sets of axioms and the primary terms is still intuitive. These sets are established on the principle of obviousness, possibly truthfulness, which provides external justification for the entire theory. In this way, the proof procedure is also established. Only accepted

19 K. Ajdukiewicz, *Pojęcie dowodu w znaczeniu logicznym*, in: idem, *Język i poznanie*, Vol. 1, Warszawa 1960.

20 Idem, *Pragmatic Logic*, Dordrecht 1974.

21 M. Tkaczyk, *Kazimierz Ajdukiewicz's Philosophy of Mathematics*, "Stud East Eur Thought" 68 (2016), pp. 21–38.

rules are allowed to be used, while any proof step, however obvious, cannot be performed if the accepted axioms and rules do not allow it. Thus, intuition remains a fundamental feature, but in place of openness, there is a closed base for theory, which is why this stage is called axiomatic intuition.

The third, or axiomatic abstract, stage, like the second, uses a closed and fixed set of axioms and primary terms. Thus, the rule of evidence procedure is also closely defined here. In contrast, the way of determining what is in these sets is completely arbitrary. This is because the primary terms of the theory are not taken in any sense as established outside of the theory. The only way to determine their meaning is to use them in axioms. At this stage, the intuitiveness and obviousness that have hitherto accompanied the base of mathematical theories disappear.

Returning to the consideration of the disagreement between Frege and Hilbert, let us note that the difference between the second intuitive axiomatic stage and the third abstract axiomatic stage is analogous to the difference between Frege's and Hilbert's respective views. Both of these authors expect mathematics to be a perfect deductive construct, free of formal defects, while only Frege claims that primary terms have an intuitive meaning, external to deductive theory. Hence, only in Frege's conceptualization, as in Ajdukiewicz's second stage, can one speak of the pure truthfulness and obviousness of the accepted axioms. Hilbert expects from the axioms only that they form the base for all theorems, and the primary terms used in them have the meaning given to them by the axioms. As such, this is how he reconstructed Euclidean geometry.[22] There, concepts like "point," "straight line," or "plane" receive precise meanings by being used in axioms. On the other hand, their intuitive meaning accepted by Euclid in *Elements* does not play any greater role in this theory besides a heuristic role. In this way, those specific constants become a sort of variables prepared for any interpretation. Despite Hilbert's declarations, a purely formal axiomatic system, free from any errors and references to an intuitive understanding of the terms used, was not obtained until the seventh edition of the aforementioned work. The style of practising mathematical theories along the lines of Hilbert's work is sometimes called the algebraic approach.[23]

3. A hypothetical *vs.* assertive point of view

According to Ajdukiewicz, deductive theories can be evaluated from the perspective of the attitudes taken towards the axioms. The axioms of a given theory can be approached in two ways. It is possible to take a hypothetical position, in which

22 D. Hilbert, *Grundlagen der Geometrie*, Leipzig1903.

23 Cf. S. Shapiro, *Categories, Structures, and the Frege-Hilbert Controversy…*, p. 66.

the axioms are treated as a kind of hypothesis, adopted to see what can be proved by employing them. It is also possible to take a position in which axioms are accepted sentences, that is, they are treated as statements asserting something about a certain reality. Therefore, two styles of doing deductive science are said to exist, the first of which is called hypothetical (hypothetico-deductive), and the second is called assertive (assertive-deductive).

The hypothetical style theories are characterized by Ajdukiewicz as follows:

> A hypothetical deductive system is a science in which at the outset we list a number of statements without adopting any attitude toward them, i.e., without either accepting or rejecting them, and next we derive from them by deduction (but do not infer) other statements that follow from the former. The statements listed at the outset (but neither accepted nor rejected) also are called axioms, and the statements derived from them (but also neither accepted nor rejected on that account) also are called derived theorems.[24]

Assertive deductive theories can be characterized as follows:

> Those deductive systems in which the axioms are asserted, and hence accepted, and in which on the strength of the acceptance of the axioms we arrive, by deductive inference, at accepting derived theorems, are called assertive deductive systems.[25]

Thus, a researcher who accepts the axioms of, for example, geometry, treats them as asserted sentences (stating something about a certain reality) is cultivating the theory in an assertive-deductive style. It can be assumed that this is how the axioms of geometry were treated by Euclid when he wrote *Elements*, or by Giovanni G. Saccheri, who negated Euclid's fifth postulate and derived several theorems of non-Euclidean geometry, assuming that this is merely an *ad absurdum* proof of the truth of this axiom. In contrast, a researcher may propose a certain geometry as an axiomatic theory, but at the same time does not treat the theory's axioms as asserted statements. In this case, they cultivate the theory in a hypothetico-deductive style. It can be assumed that non-Euclidean geometries, such as Nikolai Lobachevsky's geometry, were created in this way. He consciously questioned Euclid's fifth postulate and decided to develop a theory based on its negation, while not determining whether this postulate or its negation are true. Notice that the researcher's attitude toward the theory does not in any way affect the content of the theory, that is, the set of axioms and theorems derived from them. A deductive theory is itself a creation independent of the researcher's approach (hypothetical or assertive) toward its axioms (theorems).

24 K. Ajdukiewicz, *Pragmatic Logic…*, pp. 206–207.

25 Ibidem, p. 207.

Also of note is the situation in which deductive theory is in the third axiomatic abstract stage of development and is cultivated in an assertive style.

This stage assumes that the primary terms of a theory obtain meaning only by using them in axioms, so they do not possess it from the outside. The axioms constructed from these terms constitute their meaning within a given theory, and at the same time are treated as accepted statements. Ajdukiewicz believes that axioms "establish (their) meanings anew by deciding that the said terms are to denote such objects (i.e., individuals, classes, relations) which satisfy the axioms of a given theory, i.e., satisfy the conditions formulated in those axioms."[26] The result of this approach is the planned ambiguity of the primary terms, which manifests itself in the fact that their denotation is not fixed definitively. Finding a model for a deductive theory constructed in this way is, as it were, the next step in its construction. As such, the unambiguous denotation for the primary terms is established, and thus the axioms become true statements concerning the indicated domain. The establishment of a reference between the theory and the domain in question presupposes the existence of certain objects about which the theory speaks. Hence, a model, in reality, is always preferred over a model in another deductive theory. That is, this preference holds true in the sense that descriptive semantics is superior to formal semantics.[27] On the other hand, of the two proofs of the consistency of deductive theory, the absolute one, which establishes the meaning of primary terms so that they denote objects whose existence is undeniable, will be better.

Deductive (axiomatic abstract) theory understood in this way can be practised in an assertive style, that is, its axioms are treated as asserted statements. Thus, the question can be raised as to what they state. In the simplest terms, the axioms state something very general about a certain reality common to many domains. Advocates of mathematical structuralism will say that they describe a certain structure, which, by its nature, is prepared for further interpretation.

Practising deductive theory, or not, is a personal choice of the researcher, so the assertion of axioms, or lack thereof, must be viewed in this way. On the other hand, someone who builds deductive theories in an assertive deductive style may additionally hold certain philosophical beliefs about the nature of the object described by the theory. If we consider the structure as this object, we can classify views of this type as ontological structuralism. It will be classified as positive when this general object (structure) is considered to exist. The manner of its existence is a secondary matter at this point. In contrast, under the same assumptions, some structuralism

26 Ibidem, p. 203.

27 Cf. M. Tkaczyk, *Logika czasu empirycznego*, Lublin 2009, pp. 12–14.

can be considered to be negative, if the ontological position regarding the nature of this structure is somehow eliminative. Here again, the details of this view are unimportant.

It may also be the case that while practising assertive style theories, we suspend judgment on the nature of the object, which is the structure, and at the same time consider that only by explicitly establishing references for primary terms, we can determine the nature of these objects under discussion. In this way, establishing different scopes for primary terms can lead us to different ontologies. Thus, we treat axioms as recognized statements while we suspend judgment on the ontological nature of the reality they describe. This attitude may bring us closer to Quine's conception of global structuralism, which, according to Resnik, "is non-ontological and simply another formulation of ontological relativity."[28]

Only at this point can we try to clarify what Resnik meant when he formulated his view of non-ontological structuralism. We think that his vision of structuralism corresponds to what is behind the concept of deductive theory, which is at the third axiomatic abstract stage of development and is practised in an assertive style. At the same time, the researcher does not take a determined ontological position about a structure as an object of mathematics. In contrast to this, other types of structuralisms (ontologically positive and negative) are formed by treating mathematical theories as theories that are at the third axiomatic abstract stage and, at the same time, cultivated in an assertive style, albeit with a determined ontological position. If this ontology is positive, we have non-eliminative structuralism, while negative ontology results in eliminative structuralism.

4. Hypothetical structuralism

In light of the above distinctions, it is still necessary to consider the situation in which a mathematical theory is in the abstract axiomatic stage but is practised in the hypothetical style. This style assumes that axioms are not treated as recognized statements. This is especially the case when the theory's primary terms, which appear in its axioms, are treated as symbols of variables, without prejudging anything about their meaning. This approach does not allow us to conclude that axioms can be recognized statements. This is because primary terms, which are treated as variables, are not bound by any quantifiers, only their semantic category is determined. As such, Ajdukiewicz states, "both the axioms and the theorems derived from them are not statements, which by their nature can be true or false,

28 M. Resnik, *Non-ontological Structuralism…*, p. 307.

and become sentential schemes about which no judgment is made as to their truth or falsity."[29]

Analogues to what we have said should be sought in the standard way of introducing variables. The formula $2 + 2x = 6$ is a scheme in which the semantic category of the variable "*x*" is specified. This expression in this form does not state anything that can be accepted or rejected, and the symbol "*x*" has no specific meaning. We only know what and how we can substitute for the variable "*x*" to result in a statement that is true or false. Of course, we can also bind the variable "*x*" with a quantifier and thus obtain a true or false statement.

Note that we can now transfer this way of thinking to the following expression:

Axiom 1: $\forall x \exists y : (x < y)$,

which is one of the axioms of the elementary theory of inequality.[30] In addition to several logical symbols and the variable "*x*," this expression is built additionally from one specific primary term "<." Thus, the expression is a closed formula, which we can assume to be true or false, and it can also be seen as a schematic formula. This will happen if we treat the primary term "<" like a variable. The rationale for this fact is its planned ambiguity and, therefore, the lack of one established way to read it. Consequently, the entire statement has no single predetermined meaning. This statement may be interpreted as describing a fact from the rational number structure. In that way, the meaning of the symbol "<" is fixed as "is less than." But at the same time, we may propose another interpretation, wherein the meaning of the symbol is fixed as "lies to the left of," and the statement is about the points on a given straight line. Another possible interpretation of the symbol is "is earlier than," and then the axiom is true in the range of time moments.

Ajdukiewicz makes it clear that one can understand the axioms of deductive theories as certain kinds of schemes prepared for interpretation:

> Since […] the axioms and the derived theorems of an abstract deductive theory are not statements, but schemata of statements, hence they may be neither accepted nor rejected. Hence, in this approach, an abstract deductive theory does not consist of any-thing that could express the conviction of the researcher who is concerned with that theory. In pursuing his research he does not assert anything. His work is confined to deriving by deduction schemata of statements, called derived theorems, from schemata of statements, called axioms; the derived theorems, being not statements, but schemata of statements, also do not state anything.[31]

29 K. Ajdukiewicz, *Pragmatic Logic…*, p. 205.

30 Ibidem, p. 203.

31 Ibidem, p. 206.

Thus, it is possible to build a deductive theory based on axioms that are understood as schemes. Such a theory has the same content as a theory built on recognized expressions, that is, in the assertion style. In either case, one performs exactly this action on expressions, and it is possible to prove the same assertions. The hypothetical style results in the fact that all theorems derived from axioms, which are schemes, are also schemes. All proofs possible in this theory can be seen as certain ready-to-interpret inferential schemes.[32] These schemes are valid, that is, they will never allow one to move from a true premise to a false conclusion. Thus, a deductive theory built in the hypothetical style turns out to be a catalogue of inferences ready for use in any science.

Note that cultivating deductive theories in the hypothetical style, in its effect, produces a theory containing valid schemes of inference, while in principle, it does not produce new knowledge about some reality outside the theory. In contrast, practising deductive theory in an assertive style will be closely related to expanding the knowledge of a certain area of reality about which the axioms of the theory treat as asserted statements. Thus, a hypothesis-driven researcher does not state anything about reality outside of the theory but rather prepares correct inference schemes for interpretation. If it turns out that it is possible to interpret the primary terms in such a way as to turn the axioms as schemes into statements that are true about a certain fragment of reality, then we automatically obtain several well-founded statements about this subject.[33]

The proposal of axiomatic theory as a catalogue of valid inference schemes becomes clearer considering the distinction between two aspects of reasoning. We can evaluate each inference in terms of its formal and material correctness. An argument is formally correct if it is deductively valid. Moreover, the inference that is simultaneously formally and materially correct is sound.[34]

A materially correct inference is required to have its premises as true sentences. In principle, the evaluation of the truthfulness of the premises belongs to an expert in the respective field, so the material correctness of inferences on the grounds of mathematics is evaluated by a mathematician. The task of examining the material correctness of mathematical inferences, or proofs, is beyond the axiomatic theories, in which axioms are not treated as true sentences. What may be of interest from this point of view is to check whether any formula "*A*" and the same formula preceded by the negation symbol, "¬*A*," are derivable from

32 Ibidem, p. 110.

33 Ibidem, p. 206.

34 D. Bonevac, *Deduction: Introductory Symbolic Logic*, Malden, MA 2003, pp. 17–18.

the accepted axioms. If it turns out that there is such a pair of derivable formulas, then without evaluating the truth of any of these, we could conclude that the theory is contradictory. One of the formulas in this pair is false, and therefore, it is certain that one of the axioms is false. Of course, such a statement is possible only under the assumption that we understand the symbol "¬" as a classical functor of negation. Consequently, this leads us to the fact that standard logic, as a base theory, and a possible metatheory for the deductive theory under study, are not understood as purely formal deductive theories.

The aforementioned catalogue of correct inference schemes requires only formal correctness from subsequent proof steps. Formal correctness is nothing but validity, and as such, the construction of axiomatic and abstract hypothetical-style theories is reduced precisely to providing a catalogue of formally correct inferences. The formal correctness of the subsequent proof steps is guaranteed by appropriately selected rules of the proof procedure. These inferences, or mathematical proofs, are prepared for interpretation in the same way as inferences in logical theories. In each of these cases, the creator of such a theory does the hard formal work in creating the theory, but this is only service work when compared to all other sciences.

> For if a researcher who is studying real facts succeeds in finding out that the facts he is concerned with satisfy the axioms of a given abstract deductive theory (i.e., if the sphere of those facts is a model of that theory), then owing to the work done earlier by the scientist who studied that abstract theory by deducing derived theorems from axioms, the student of facts can learn, without any extra effort on his part, that the domain he is concerned with also satisfies the derived theorems of that theory; he thus signally broadens his knowledge of the sphere of facts he is studying.[35]

By that means, researchers studying deductive theories create a constantly expanding catalogue of valid inference schemes, which, by definition, are prepared for especially understood interpretation. Thus, each deductive theory in the abstract axiomatic stage and the hypothetical style, both one by one and together, form structures that describe the relations between certain objects. These relations are indicated in no other way than by their description contained in the axioms and derivative theorems of a given theory.

Symptomatic of this approach are not ambiguity and underdetermination of the primary terms since these also occur within the assertive style. The ambiguity and underdetermination of primary terms are appropriate for the third abstract stage of the development of deductive science, according to Ajdukiewicz.

35 K. Ajdukiewicz, *Pragmatic Logic…*, p. 206.

For the hypothetical style approach, it is appropriate to suspend judgment on the reality described by the mathematical theory. Nothing is assumed about this reality, so it can be said without risking that the axioms and theorems are not treated as recognized statements. This approach consistently does not allow us in any way to consider that structures (understood this way or that way) are objects studied and described by mathematicians. The mathematical structure is understood here in a different way. The structure is formed by axioms, accepted as hypotheses and theorems, which can be derived from these axioms by established rules of proof.

Thus, the mathematical structure is understood as the structure of a deductive theory, in which attention is first paid to the relations between the expressions of the theory. Since such a theory is practised in a hypothetical style, that is, if we do not accept the axioms, we cannot treat the theory as describing some reality. This hypothetical approach to deductive theories, as theories in the third axiomatic abstract stage, is called hypothetical structuralism.

The hypothetical structuralism proposed here is non-ontological, just like Resnik's concept discussed above. Judgment on the existence and nature of mathematical structures is suspended here, just as Resnik does in non-ontological structuralism. In contrast, the fundamental difference becomes evident in the way axioms (and theorems) are treated as terms stating something about a certain reality. Resnik's non-ontological structuralism is a deductive theory cultivated in an assertive style, while hypothetical structuralism is a deductive theory in a hypothetical style.

Let's emphasize it again, the style of practising a dedicatory theory does not affect its content. As such, we can say that as far as the mathematical content is concerned, it is neutral. Returning to axiom 1 of the elementary theory of inequality presented above; this theory has seven different axioms. The only primary term of this theory is “<,” the meaning of which is established precisely in these seven axioms since it is a deductive theory in the third axiomatic abstract stage. The other terms used in the axioms are logical terms with an established meaning in logic. The theorems of this theory derived from the axioms describe a certain reality in the context of this one primary term.

A proponent of Resnik's non-ontological structuralism adopts an assertive perspective, that is, they accept the axioms and the theorems derived from them. Likewise, with an advocate of hypothetical structuralism, they too accept the axioms as the base for the theory and accept the derivation of all possible theorems. In this sense, regardless of the position taken, the content of the deductive theory is identical.

The difference is seen elsewhere. Resnik's non-ontological structuralism accepts that the above axiom says something about a certain reality, in this case, mathematical structures, which we can call the structure of inequality relations. At the same time, it does not make a statement about how this structure exists and what its ontological nature is.

The acceptance of axioms and theorems has the effect of treating a deductive theory as having a certain object of study, which is independent of the theory and what the theory describes. This object for Resnik is what we can call "the structure of inequality relations." Of course, non-ontological structuralism suspends judgment on how this structure exists and its ontological nature. The opposite is true of hypothetical structuralism, which does not apply the moment of assertion to the axioms of the theories under study. Its setting is hypothetical, so the axioms (and theorems), in this case, are not accepted sentences. The above axiom 1 of the theory of inequality can be regarded as describing a certain hypothesis, on the base of which the theorems of the theory are derived. The whole theory turns out to be ultimately a set of hypotheses derived from the hypothetical axioms. Of course, the axioms are taken completely arbitrarily. There is also no place here for questions about the existence and nature of mathematical objects, but the prohibition is stronger than in the case of Resnik's concept. It is not as if the judgment about the ontology of mathematical structures is suspended, and we consciously do not answer it. In the hypothetical conception, it is forbidden to pose this kind of question at all because, by definition, a mathematical theory constructed as such does not describe anything that can be called a structure, or rather, it does not describe any object at all. The possible relevance of the theory to a certain fragment of reality is something that we could consider to be beyond the scope of a theory practised in this way.

What we mentioned about the elementary theory of inequality can be said about any other axiomatic mathematical theory. In Resnik's non-ontological structuralism, the axiomatic arithmetic of natural numbers will be seen as describing the structure of natural numbers, but the nature of this structure will not be specified. Going further, one can consider two contradictory mathematical theories, say, Euclidean geometry and non-Euclidean geometry. Their axioms can be thought of as accepted statements that state something about certain structures. Each of these structures, both Euclidean and non-Euclidean, can exist in one way or another, and we can study the interrelationships between them and their properties, but the ontologies are beyond the reach of this view.

Hypothetical structuralism, on the other hand, focuses only on the construction of theory. For this, a deductive theory is an object of interest and a structure is a subject to study. The only tool a mathematician has is the proof, so from any axioms by proof, theorems are derived, which form a well-defined structure. Based on the fact that the axioms are only hypotheses, the entire theory, and therefore all theorems, are also hypothetical. Of course, axioms, and consequently theorems, describe certain relations between objects. In the elementary theory of inequalities, relations between objects are described, and these can serve as models for the theory. The same is true in the axiomatic view of the arithmetic of natural numbers, or any axiomatically constructed geometry: each theory says only as much as is said in the axioms. Moreover, these axiomatic theories create a separate, unique structure of interrelationships between axioms and theorems. The relation that connects all expressions is the relation of derivability. What is central, from the perspective of hypothetical structuralism, is the construction of a correct deductive theory, not the accurate representation or description in the theory of some structure that is outside the theory.

Once again, the interpretation referred to here, i.e., establishing that the symbol “<“ is understood in one way or another (e.g., the expression “$x < y$” reads “x is before y”), is the next step in the construction of a mathematical theory. The transition apart from the perspective of hypothetical structuralism is not necessary, but it enriches the research being done. Many researchers will say that theories for which interpretations have been found in reality are better than those for which no such interpretation exists. Still, others will say that finding an interpretation for a mathematical theory in another mathematical theory (e.g., reading the expression “$x < y$” as “the natural number x is smaller than the natural number y”) is also a momentous discovery.

In summary, hypothetical structuralism is called structuralism, but in a different sense from all the others, including Resnik’s non-ontological structuralism. It does not claim that mathematical theories describe objects called structures; instead, within this position, it is only believed that mathematical theories themselves create certain structures. The arbitrarily accepted axioms and theorems derived from them in a strict deductive method form a structure of interrelated expressions. What connects them is the relation of derivability. Therefore, the mathematician’s job is to build theories of this kind, that is, to create or discover structures to be understood. The same work is done by a mathematician who is a follower of Resnik’s concept of non-ontological structuralism. In each case, theories have the same content, and the proof is the only tool for justifying theorems. Hypothetical structuralism is free from the assumption that the objects studied by mathematics are structures independent of mathematical theories.

Conclusion

The question of the hypothetical nature of Resnik's conception of non-ontological structuralism has proved to be legitimate. Resnik's shift in views regarding the ontology of mathematical structures was not clear and needed to be explored further. It was possible to understand what Resnik's declared suspension of judgment on the ontology of mathematical structures meant. Several distinctions proved to be helpful in this regard.

We showed that the third axiomatic abstract stage of the development of deductive theories, proposed by Ajdukiewicz, is appropriate for understanding mathematics as a science of structures. Any mathematical theory can be practised in one of two styles, hypothetical or assertive. The assertive style turns out to be appropriate for Resnik's non-ontological structuralism. We believe that these views can be understood in the way that mathematics is a deductive science, being in the abstract axiomatic stage, practised in the assertive style, which at the same time suspends its judgment regarding the ontological nature of structures as an object of study. We consider this explanation clearer and better than what can be found in Resnik's work.[36]

The distinctions introduced also helped formulate the position of hypothetical structuralism. This view holds that mathematics is a deductive science when in the abstract axiomatic stage, cultivated in a hypothetical style. The question of the ontology of mathematical structures within this conception is not posed at all, primarily because the concept of structure has a different meaning. Structure within this conception is understood as a hierarchy of interrelated relations of the derivation of expressions that make up a deductive theory.

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IS NON-ONTOLOGICAL STRUCTURALISM HYPOTHETICAL?

Summary

Michael Resnik, the founder of modern structuralism in the philosophy of mathematics, changed his views and proposed a new non-ontological structuralism. Resnik is considered a prominent figure in modern structuralism within the realm of contemporary philosophy of mathematics, and his *sui generis* structuralism is regarded as one of the most significant and frequently discussed positions in the field. This article examines the motivations behind Resnik's change of perspective. His new position is presented in detail, and an attempt is made to contrast it with selected views in the philosophy of mathematics. The discussion is conducted in the context of the Frege-Hilbert Controversy, which centers on the status of mathematical theory axioms and the meaning attributed to primary terms. The three-stage concept of the development of deductive sciences, proposed by Kazimierz Ajdukiewicz, is also introduced. This concept, inspired by Hilbert's ideas, outlines three stages in the evolution of deductive theories:

(1) pre-axiomatic deductive, (2) axiomatic deductive, (3) abstract axiomatic. Each of these stages possesses unique characteristics that illuminate the nature of deductive theories, particularly in relation to the approach to sets of axioms and primary terms within these theories. Additionally, two styles of practicing deductive theories (assertive and hypothetical) are discussed. Ultimately, an exploration of Resnik's non-ontological structuralism provides insight into how this novel structuralist concept should be understood and clarifies the author's actual claims. Alongside these distinctions, a new conception of hypothetical structuralism, distinct from non-ontological structuralism, is formulated. This novel structuralism is rooted in the hypothetical approach to practicing deductive theories.

Keywords: philosophy of mathematics, Resnik, non-ontological structuralism, hypothetical structuralism

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